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The support of the momentum density of the Camassa–Holm equation

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Bounds for the size of the support of a compactly supported momentum density of the Camassa–Holm equation are derived. This is achieved by estimating the first Dirichlet eigenvalue of the support. This elaborates the result on the preservation of its compactness, and gives more information on the velocity by estimating the size of the region where it is not that well understood.

1. Introduction

The initial value problem of the Camassa–Holm equation is

$$\begin{cases} u_t - u_{xxt} + 3uu_x = 2u_xu_{xx} + uu_{xxx}, & x \in \mathbb{R}, \quad t > 0, \\ u(0, x) = u_0(x) & x \in \mathbb{R}. \end{cases} \tag{1.1}$$

Let $m(t, x) = u(t, x) - u_{xx}(t, x)$ be the momentum density of u. The problem in the momentum density form is

$$\begin{cases} m_t + um_x + 2u_xm = 0, & x \in \mathbb{R}, \quad t > 0, \\ m(0, x) = m_0(x) & x \in \mathbb{R}. \end{cases} \tag{1.2}$$

From [1,2], if $u_0 \in H^3(\mathbb{R})$, then there is a $T^* = T^*(u_0) > 0$ such that for all $T \in [0, T^*)$, (1.1) has a unique strong solution $u \in C([0, T]; H^3(\mathbb{R})) \cap C^1([0, T]; H^2(\mathbb{R}))$, with the stability property that $u_0 \to u(\cdot, u_0)$ is a continuous map from $H^3(\mathbb{R})$ to $C([0, T]; H^3(\mathbb{R})) \cap C^1([0, T]; H^2(\mathbb{R}))$. In that case, $m \in C([0, T]; H^1(\mathbb{R})) \cap C^1([0, T]; L^2(\mathbb{R}))$ is a strong solution to (1.2).

It is known that for the Camassa–Holm equation, the compactly supported initial momentum density $m_0 = u_0 - u_{0,xx}$ will remain compactly supported [3], though the same statement for u is false [3–5]. Also, the exponential behavior of u in x outside $\operatorname{supp} m$ is obvious. For $s > 5/2$, a further detailed description of u outside $\operatorname{supp} m$ is given in [6, Theorem 4]. In this article, we estimate the size of $\operatorname{supp} m(t, \cdot)$. This is a further elaboration of the compactness of $\operatorname{supp} m(t, \cdot)$, and gives more information on u by estimating the size of the region where u is not that well understood.

The approach is inspired by the work of Kim [7], where the geometric properties of a vortex patch under the Euler flow are estimated by estimating the Dirichlet eigenvalues of the support of the patch, exploiting the relations between the eigenvalues and the geometric properties of a domain. The momentum density of the Camassa–Holm equation is analogous to the vortex of an Euler flow. They satisfy similar first order nonlocal nonlinear equations, and determine the velocities via similar integrals. There are fundamental differences though. The Euler flow is measure preserving but the Camassa–Holm flow is not, as it is derived from approximating the Euler equation. Vortices are propagated along the two-dimensional Euler flow but momentum densities are stretched. Nonetheless, these differences can be overcome and an estimate of the size of $\operatorname{supp} m(t, \cdot)$ is given below.

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Theorem 1.1. *Let $m \in C([0,T]; H^1(\mathbb{R})) \cap C^1([0,T]; L^2(\mathbb{R}))$ be a strong solution of (1.2). For $t \in [0,T]$, let $D(t)$ be the support of $m(t,\cdot)$. Suppose that $D(0)$ is connected. Suppose a constant M is such that $|u_x| \le M$ on $[0,T] \times \mathbb{R}$. Then*

(a)

$$|D(0)|e^{-2Mt} \le |D(t)| \le |D(0)|e^{2Mt}. \tag{1.3}$$

(b) *M can be taken to be $e^{3KT/2}\|m_0\|_{L^2(\mathbb{R})}/2$, where $K > 0$ is a constant such that $u_x(t,x) \ge -K$ for $(t,x) \in [0,T] \times \mathbb{R}$.*

(c) *If $m_0(\cdot) \in H^1(\mathbb{R})$ does not change sign, or if there is an $x_0 \in \mathbb{R}$ such that*

$$m_0(x) \begin{cases} \le 0 & \text{on } (-\infty, x_0] \\ \ge 0 & \text{on } [x_0, \infty), \end{cases} \tag{1.4}$$

then a global strong solution to (1.2) exists [1]. In case $m_0 \in H^1(\mathbb{R}) \cap L^1(\mathbb{R})$, (1.3) holds for all $t \in [0,\infty)$ and M can be taken to be $\|m_0\|_{L^1(\mathbb{R})}/2$.

Remark: (a) Since in this case $u \in C([0,T]; H^3(\mathbb{R}))$, the Sobolev Embedding Theorem implies that $|u|$ is bounded by some $L > 0$ on $[0,T] \times \mathbb{R}$. As the support of $m(t,\cdot)$ propagates along the flow (see (2.2)), we have easily the linear upper estimate $|D(t)| \le |D(0)| + 2Lt$. As the corresponding lower estimate $|D(t)| \ge |D(0)| - 2Lt$ is not informative for $t \ge |D(0)|/2L$, the lower bound in Theorem 1.1 is probably more valuable.

(b) The Camassa–Holm equation only approximates the incompressible Euler flow [8]. Hence the size of the supp $m(t,\cdot)$ can change, even though the support propagates along the flow. On the other hand, as it is an approximation of an incompressible flow, we expect that $|D(t)|$ changes only a little. Nonetheless, Theorem 1.1 still provides a lower bound estimate for $|D(t)|$.

(c) For supp $m(0,\cdot)$ not connected, one can apply the reasoning to its convex hull or its connected components.

(d) Similar result and proof holds for the Degasperis–Procesi equation and other more general shallow water equations, for example those studied in [9].

2. Proof of the main theorem

We fix our notations and recall a few facts to be used. Let $u \in C([0,T]; H^3(\mathbb{R})) \cap C^1([0,T]; H^2(\mathbb{R}))$ be a strong solution to (1.1). Let $s \in [0,T]$. Let $\eta(t;\alpha,s)$ be the solution of

$$\begin{cases} \dfrac{d\eta(t;\alpha,s)}{dt} = u(s+t, \eta(t;\alpha,s)), & s, s+t \in [0,T],\ \alpha \in \mathbb{R} \\ \eta(0;\alpha,s) = \alpha & \alpha \in \mathbb{R}. \end{cases} \tag{2.1}$$

It is the flow with initial velocity $u(s,\cdot)$. Then $\eta(t;\cdot,s) : \mathbb{R} \to \mathbb{R}$ is an increasing diffeomorphism. It is shown [1,3] that

$$m(t, \eta(t;x,0))\eta_x^2(t;x,0) = m(0,x), \tag{2.2}$$

which implies that supp m propagates along the flow. Let $\psi \in L^2(D(s))$. For $s+t \in [0,T]$, let $\psi^t \in L^2(D(s+t))$ be given by

$$\psi^t(\eta(t;\alpha,s)) = \psi(\alpha). \tag{2.3}$$

Let $\Omega \subset \mathbb{R}$ be an open interval. Let $\lambda_1(\Omega)$ be the first Dirichlet eigenvalue of the Laplacian on Ω. Then

$$\lambda_1(\Omega) = \inf_{\substack{\phi \in H_0^1(\Omega) \\ \|\phi\|_{L^2(\Omega)}=1}} \|\phi'\|^2_{L^2(\Omega)}. \tag{2.4}$$

It is just $(\pi/|\Omega|)^2$ and the normalized eigenfunctions are the suitable translates of

$$\pm\left(\frac{2}{|\Omega|}\right)^{1/2} \sin\frac{\pi x}{|\Omega|}.$$

Lemma 2.1. *Let $s, s+t \in [0,T], \alpha \in D(s)$ and $\psi \in H_0^1(D(s))$ ($L^2(D(s))$ is enough for (c)). Then under the hypothesis of Theorem 1.1,*

(a)

$$e^{-M|t|} \le \eta_\alpha(t;\alpha,s) \le e^{M|t|}, \tag{2.5}$$

(b)

$$|\psi'(\alpha)|e^{-M|t|} \le |(\psi^t)'(\eta(t;\alpha,s))| \le |\psi'(\alpha)|e^{M|t|}, \quad \textit{and} \tag{2.6}$$

(c)

$$e^{-M|t|/2}\|\psi\|_{L^2(D(s))} \le \|\psi^t\|_{L^2(D(s+t))} \le e^{M|t|/2}\|\psi\|_{L^2(D(s))}. \tag{2.7}$$

Proof. (a) Differentiate (2.1) with respect to α. Use the hypothesis $|u_x| \le M$ on $[0, T] \times \mathbb{R}$, noticing that $\eta_\alpha(t; \alpha, s) > 0$ [1], to get

$$-M\eta_\alpha(t; \alpha, s) \le \eta_{\alpha t}(t; \alpha, s) \le M\eta_\alpha(t; \alpha, s). \tag{2.8}$$

This implies the conclusion.

(b) Differentiate (2.3) with respect to α to get

$$(\psi^t)'(\eta(t; \alpha, s))\eta_\alpha(t; \alpha, s) = \psi'(\alpha).$$

The conclusion follows from (a).

(c) Use the change of variable $x = \eta(t; \alpha, s)$, (2.3) and (a) to get

$$\|\psi^t\|^2_{L^2(D(s+t))} = \int_{D(s+t)} \psi^t(x)^2 \mathrm{d}x = \int_{D(s)} \psi^t(\eta(t; \alpha, s))^2 \eta_\alpha(t; \alpha, s)\mathrm{d}\alpha \begin{cases} \le e^{M|t|} \int_{D(s)} \psi(\alpha)^2 \mathrm{d}\alpha \\ \ge e^{-M|t|} \int_{D(s)} \psi(\alpha)^2 \mathrm{d}\alpha. \end{cases} \quad \square$$

Lemma 2.2. *Under the hypothesis of Theorem 1.1, for $s, s + t \in (0, T)$,*

$$\limsup_{t\to 0+} \frac{\lambda_1(D(s+t)) - \lambda_1(D(s))}{t} \le 4\lambda_1(D(s)), \tag{2.9}$$

$$\liminf_{t\to 0-} \frac{\lambda_1(D(s+t)) - \lambda_1(D(s))}{t} \ge -4\lambda_1(D(s)), \tag{2.10}$$

Proof. Let $t > 0$. Let $\phi_1 \in H^1_0(D(s))$, $\|\phi_1\|_{L^2(D(s))} = 1$, be a first normalized eigenfunction on $D(s)$. Then

$$\begin{aligned} \lambda_1(D(s+t)) - \lambda_1(D(s)) &= \inf_{\substack{\psi \in H^1_0(D(s+t)) \\ \|\psi\|_{L^2(D(s+t))}=1}} \|\psi'\|^2_{L^2(D(s+t))} - \|\phi_1'\|^2_{L^2(D(s))} \\ &\le \|\phi_1^t\|^{-2}_{L^2(D(s+t))}\|(\phi_1^t)'\|^2_{L^2(D(s+t))} - \|\phi_1'\|^2_{L^2(D(s))}, \end{aligned} \tag{2.11}$$

From the right halves of (2.5)–(2.7),

$$\begin{aligned} \|\phi_1^t\|^{-2}_{L^2(D(s+t))}\|(\phi_1^t)'\|^2_{L^2(D(s+t))} &= \|\phi_1^t\|^{-2}_{L^2(D(s+t))} \int_{D(s)} [(\phi_1^t)'(\eta(t; \alpha, s))]^2 \eta_\alpha(t; \alpha, s)\mathrm{d}\alpha \\ &\le \|\phi_1^t\|^{-2}_{L^2(D(s+t))} e^{3Mt}\|\phi_1'\|^2_{L^2(D(s))} \le e^{4Mt}\|\phi_1'\|^2_{L^2(D(s))}. \end{aligned} \tag{2.12}$$

(2.11) and (2.12) give

$$\limsup_{t\to 0+} \frac{\lambda_1(D(s+t)) - \lambda_1(D(s))}{t} \le \limsup_{t\to 0+} \left(\frac{e^{4Mt} - 1}{t}\right)\|\phi_1'\|^2_{L^2(D(s))} = 4M\lambda_1(D(s)),$$

which is (2.9). For (2.10), let $t < 0$ and again use the right halves of (2.5) to (2.7) to get

$$\lambda_1(D(s+t)) - \lambda_1(D(s)) \le e^{-4Mt}\|\phi_1'\|^2_{L^2(D(s))} - \|\phi_1'\|^2_{L(D(s))}.$$

Consequently

$$\liminf_{t\to 0-} \frac{\lambda_1(D(s+t)) - \lambda_1(D(s))}{t} \ge \liminf_{t\to 0-} \left(\frac{e^{-4Mt} - 1}{t}\right)\|\phi_1'\|^2_{L^2(D(s))} = -4M\lambda_1(D(s)). \quad \square$$

Lemma 2.3. *Under the hypothesis of Theorem 1.1, for $s, s + t \in (0, T)$,*

$$\liminf_{t\to 0+} \frac{\lambda_1(D(s+t)) - \lambda_1(D(s))}{t} \ge -4\lambda_1(D(s)), \tag{2.13}$$

$$\limsup_{t\to 0-} \frac{\lambda_1(D(s+t)) - \lambda_1(D(s))}{t} \le 4\lambda_1(D(s)), \tag{2.14}$$

Proof. The proof is not entirely symmetric to that of Lemma 2.2. Let ϕ_1 be as in Lemma 2.2. Let $\phi_2 \in L^2(D(s))$ be such that its t-transport is a normalized first eigenfunction on $D(s+t)$: $\|\phi_2^t\|_{L^2(D(s+t))} = 1$ and $\lambda_1(D(s+t)) = \|(\phi_2^t)'\|^2_{L^2(D(s+t))}$. For $t > 0$, use the left halves of (2.5) and (2.6), and then the right half of (2.7) to get

$$\begin{aligned}\|(\phi_2^t)'\|^2_{L^2(D(s+t))} &= \int_{D(s+t)} [(\phi_2^t)'(x)]^2 \mathrm{d}x \\ &= \int_{D(s)} [(\phi_2^t)'(\eta(t;\alpha,s))]^2 \eta_\alpha(t;\alpha,s)\mathrm{d}\alpha \\ &\geq \mathrm{e}^{-3Mt} \int_{D(s)} [\phi_2'(\alpha)]^2 \mathrm{d}\alpha \\ &= \mathrm{e}^{-3Mt} \|\phi_2\|^2_{L^2(D(s))} \|(\phi_2/\|\phi_2\|_{L^2(D(s))})'\|^2_{L^2(D(s))} \\ &\geq \mathrm{e}^{-3Mt}\mathrm{e}^{-Mt} \|\phi_2^t\|^2_{L^2(D(s+t))} \lambda_1(D(s)) = \mathrm{e}^{-4Mt}\lambda_1(D(s)).\end{aligned}$$

Hence

$$\liminf_{t\to 0+} \frac{\lambda_1(D(s+t)) - \lambda_1(D(s))}{t} \geq \liminf_{t\to 0+} \frac{\mathrm{e}^{-4Mt}-1}{t}\lambda_1(D(s)) = -4M\lambda_1(D(s)).$$

The proof of (2.14) is similar. $\square$

Proof of Theorem 1.1. (a) From Lemmas 2.2 and 2.3, $\lambda_1(D(s))$ is Lipschitz and hence differentiable almost everywhere. The inequalities in the lemmas give

$$-4M\lambda_1(D(s)) \leq \frac{\mathrm{d}}{\mathrm{d}s}\lambda_1(D(s)) \leq 4M\lambda_1(D(s)),$$

which implies

$$\mathrm{e}^{-4Ms}\lambda_1(D(0)) \leq \lambda_1(D(s)) \leq \mathrm{e}^{4Ms}\lambda_1(D(0)).$$

As $\lambda_1(D(s)) = \pi^2/|D(s)|^2$, we get the desired conclusion.

(b) Recall that

$$u(t,x) = \frac{1}{2}\int_{\mathbb{R}} \mathrm{e}^{-|x-y|} m(t,y)\mathrm{d}y, \qquad u_x(t,x) = \frac{1}{2}\int_{\mathbb{R}} \mathrm{sgn}(y-x)\mathrm{e}^{-|x-y|} m(t,y)\mathrm{d}y. \tag{2.15}$$

Also, if the strong solution $u \in C([0,T];H^3(\mathbb{R})) \cap C^1([0,T];H^2(\mathbb{R}))$ exists, then $u_x \geq -K$ for some constant $K > 0$ on $[0,T]\times\mathbb{R}$, as a strong solution blows up only in the form of wave breaking [1,10]. Hence using (1.2) and integration by parts, we get [1]

$$\frac{\mathrm{d}}{\mathrm{d}t}\|m(t,\cdot)\|^2_{L^2(\mathbb{R})} = -3\int_{\mathbb{R}} u_x(t,x)m(t,x)^2\mathrm{d}x \leq 3K\|m(t,\cdot)\|^2_{L^2(\mathbb{R})}. \tag{2.16}$$

Then (2.15), Cauchy–Schwarz and (2.16) imply that for $t \in [0,T]$,

$$|u_x(t,x)| \leq \frac{1}{2}\|m(t,\cdot)\|_{L^2(\mathbb{R})} \leq \frac{1}{2}\mathrm{e}^{3KT/2}\|m_0\|_{L^2(\mathbb{R})}.$$

(c) If $m_0(\cdot) \in H^1(\mathbb{R})$ does not change sign, then (2.15) and the conservation of $\int_{\mathbb{R}} m(t,y)\mathrm{d}y$ [11] imply that for $t \geq 0$,

$$|u_x(t,x)| \leq \frac{1}{2}\|m(t,\cdot)\|_{L^1(\mathbb{R})} = \frac{1}{2}\|m_0\|_{L^1(\mathbb{R})}. \tag{2.17}$$

Next, suppose that (1.4) holds. We claim that (2.17) holds with the last equality replaced by a $\leq$ sign. It suffices to prove that for $t > 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|m(t,\cdot)\|_{L^1(\mathbb{R})} \leq 0. \tag{2.18}$$

For this, denote the function $\eta(t;x,s)$ in (2.1) by $\eta(t;x)$ when $s=0$. Notice that (1.4) and (2.2) imply

$$m(t,\eta(t,x)) \begin{cases} \leq 0 & \text{on } (-\infty, \eta(t,x_0)] \\ \geq 0 & \text{on } [\eta(t,x_0), \infty). \end{cases} \tag{2.19}$$

Now write

$$\int_{\mathbb{R}} |m(t,y)|\mathrm{d}y \quad = -\int_{-\infty}^{\eta(t,x_0)} + \int_{\eta(t,x_0)}^{\infty} m(t,y)\mathrm{d}y.$$

Differentiate with respect to t, using (1.2) and (2.1), that $H^1(\mathbb{R})$ functions vanish at $\pm\infty$, and then $m = u - u_{xx}$ to simplify the integrals, we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}|m(t,y)|\mathrm{d}y &= \int_{-\infty}^{\eta(t,x_0)} - \int_{\eta(t,x_0)}^{\infty}(um_x + 2u_x m)\mathrm{d}y - 2m(t,\eta(t,x_0))u(t,\eta(t,x_0))\\ &= \int_{-\infty}^{\eta(t,x_0)} - \int_{\eta(t,x_0)}^{\infty} u_x m\mathrm{d}y\\ &= u^2(t,\eta(t,x_0)) - u_x^2(t,\eta(t,x_0)).\end{aligned} \tag{2.20}$$

From (2.15) and (2.19), $u_x(t,\eta(t,x_0)) \geq |u(t,\eta(t,x_0))|$, which together with (2.20) gives (2.18). The proof is finished. □

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